B.Math.Hons. IInd year
IInd Midsemestral exam 2004

Algebra IV : B.Sury

Note that there are SIX problems in all to be solved.
Do 1 or 2, 3 or 4, 5 or 6.
Do THREE out of 7,8,9,10.

1. Let M, N be left A-modules and suppose N is semisimple. If $\alpha, \beta : M \to N$ are in $\mathrm{Hom}_A(M, N)$ such that Ker $\alpha \subseteq$ Ker β, show that there exists $\theta \in \mathrm{End}_A(N)$ satisfying $\beta = \theta \circ \alpha$.

2. Let A be any commutative ring and let G be a finite group. Show that the group ring $A[G]$ is left Noetherian (that is, any ascending chain of left ideals is finite) if, and only if, it is right Noetherian. You may use the map $\sum_g a_g g \mapsto \sum_g a_g g^{-1}$.

3. Let G be a finite group and $f, g : G \to \mathbb{C}$ be class functions. Prove Plancherel's formula: $< f, g >= \sum_{i=1}^{s} < f, \chi_i >< \chi_i, g >$ where $\chi_1, \ldots, \chi_s$ are the irreducible characters of G.

4. Consider the following character table of a finite group (where $\omega = e^{2\pi i/3}$) :

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
χ_{ρ_1}	1	1	1	1	1	1	1
χ_{ρ_2}	1	1	1	ω^2	ω	ω^2	ω
χ_{ρ_3}	1	1	1	ω	ω^2	ω	ω^2
χ_{ρ_4}	2	-2	0	-1	-1	1	1
χ_{ρ_5}	2	-2	0	$-\omega^2$	$-\omega$	ω^2	ω
χ_{ρ_6}	2	-2	0	$-\omega$	$-\omega^2$	ω	ω^2
χ_{ρ_7}	3	3	-1	0	0	0	0

Find the order of the group and cardinalities of the conjugacy classes.

5. Prove that every simple ring must be of the form $M_n(D)$ for some division ring D and some n.

6. Let A be a left Artinian ring (that is, every decreasing chain of left ideals is finite). If the Jacobson radical $\mathrm{Jac}(A)$ (the intersection of all maximal left ideals) is zero, prove that A is left semisimple.

7. If a finite group has exactly three irreducible complex representations, prove that it is isomorphic either to $\mathbb{Z}/3\mathbb{Z}$ or to S_3.

8. Let $G \subseteq GL_n(\mathbb{C})$ be a finite group and $r \geq 1$ be such that $\sum_g (tr(g))^r = 0$. Prove that $\sum_g g_{11}^r = 0$ where g_{11} is the $(1,1)$-th entry of g. You may use the fact that the character of $\underbrace{\rho \otimes \cdots \otimes \rho}$ is χ_ρ^r for any representation ρ.

9. Let G be a finite group and let $\bar{\rho} : G \to PGL_n(\mathbb{C}) = GL_n(\mathbb{C})/Z$ be a projective representation, that is, a homomorphism where $Z = \{tI : t \in \mathbb{C}^*\}$, the centre of $GL_n(\mathbb{C})$. Show that there exists a function $\rho : G \to GL_n(\mathbb{C})$ and a function $\alpha : G \times G \to \mathbb{C}^*$ such that $\rho(x)\rho(y) = \alpha(x,y)\rho(xy)$ for all $x, y \in G$. Further, if there exists a 'lift' of $\bar{\rho}$ to a representation $\tilde{\rho} : G \to GL_n(\mathbb{C})$, that is, $\bar{\rho}$ is the composite of $\tilde{\rho}$ and the natural homomorphism from $GL_n(\mathbb{C})$ to $PGL_n(\mathbb{C})$, prove that there exists a function $\theta : G \to \mathbb{C}^*$ such that

$$\alpha(x,y) = \theta(x)\theta(y)(\theta(xy))^{-1}.$$

10. Let G_1, G_2 be finite groups having the same numbers of elements of any given order. Call n, the common order of G_1, G_2 and consider G_1, G_2 as subgroups of S_n by the corresponding left regular representations.
(i) Prove that any two elements in $G_1 \cup G_2$, with the same order, are conjugate in S_n.
(ii) Prove that each conjugacy class in S_n intersects both G_1 and G_2 in the same number of elements.
(iii) Conclude that the representations of S_n on the sets S_n/G_1 andS_n/G_2 of left cosets, are equivalent, i.e., they have the same character.